Chapter 2 Limits and continuity

1. Limits of a function

Limit is a fundamental concept of calculus. We define the **limits of a function** in the following informal yet intuitive way. When a is a number and f is a function that is well-defined **near** a (but is not necessarily well-defined at a), we can talk about the "limit of $f(x)$ as x tends to a".

Definition 2.1 (Informal definition of limit) Let a be a real number and f be a function which is well-defined on an open interval that contains a, except possibly at a. If the number $f(x)$ gets closer and closer to a unique real number L when x gets closer and closer to a but $x \neq a$, then L is called the ***limit of $f(x)$ as x tends to a***. In symbols we write

"$f(x) \to L$ as $x \to a$",

or more compactly,

$$\lim_{x\to a} f(x) = L.$$

Note that such a limit L may or may not exist.

Example 2.2 Let a be a real number. It is obvious that

$$\lim_{x\to a} x = a.$$

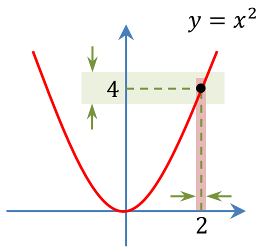
Example 2.3 Evaluate each of the following limits, if it exists.

(a) $\displaystyle\lim_{x\to 2} x^2$ (b) $\displaystyle\lim_{x\to 2} \frac{1}{x-2}$ (c) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2}$

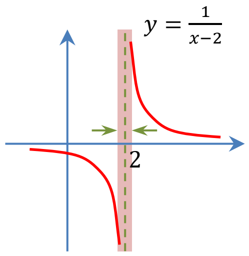
Solution:

(a) When x gets closer and closer to 2 but $x \neq 2$, the number x^2 gets closer and closer to 4. So

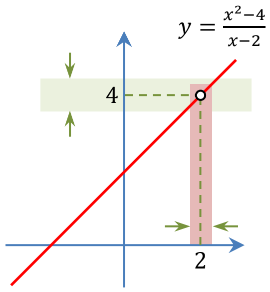
$$\lim_{x\to 2} x^2 = 4.$$



(b) When x gets closer and closer to 2 but $x \neq 2$, the number $\frac{1}{x-2}$ either becomes a very large positive number or a very large negative number. It is not always getting closer and closer to any particular real number. Therefore $\lim_{x\to 2} \frac{1}{x-2}$ does not exist.



(c) Let $f(x) = \frac{x^2-4}{x-2}$. Then <u>for every $x \neq 2$</u>, we have



$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2.$$

Now as x gets closer and closer to 2 **but $x \neq 2$**, the number $f(x)$ gets closer and closer to 4. Therefore

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

[Note that although $f(2)$ is undefined, we don't care.]

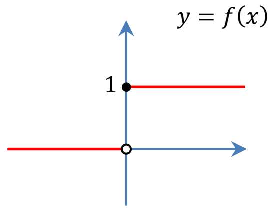
Example 2.4 Let $f: \mathbb{R} \to \mathbb{R}$ be the ***Heaviside step function*** which is defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

Evaluate the limit $\lim_{x\to 0} f(x)$ if it exists.

Solution:

No matter how close x gets to 0 and $x \neq 0$, $f(x)$ is sometimes 0 and sometimes 1. So $f(x)$ is not always getting closer and closer to a unique real number. Therefore



$$\lim_{x\to 0} f(x)$$

does not exist.

Remark 2.5 It is important to identify the difference between the two statements

$$\lim_{x\to a} f(x) = L \qquad \text{and} \qquad f(a) = L.$$

The key difference is that

- $\lim_{x\to a} f(x)$ is about the values of f when the input is **close to but not equal to** a, whereas
- $f(a)$ is the value of f when the input is **exactly equal to** a.

So when computing $\lim_{x\to a} f(x)$, we do not care about the value $f(a)$ at all. $f(a)$ may be

- well-defined ($f(2) = 4$ as in Example 2.3 (a) or $f(0) = 1$ as in Example 2.4), or
- undefined (as in Example 2.3 (b) and (c));

but it is irrelevant anyway. When computing the limit we focus only on those values $f(x)$ when x is **near** a **but not equal to** a. That is why we are allowed to cancel common factors $(x-a)$ appearing in both the numerator and the denominator of $f(x)$, as in Example 2.3 (c).

The limit of a function at a number is **unique** by the informal definition.

Theorem 2.6 (Uniqueness of limit) *Let a be a real number and f be a function. If real numbers L and M are both limits of $f(x)$ as x tends to a, then $L = M$.*

Example 2.7 Evaluate the limit

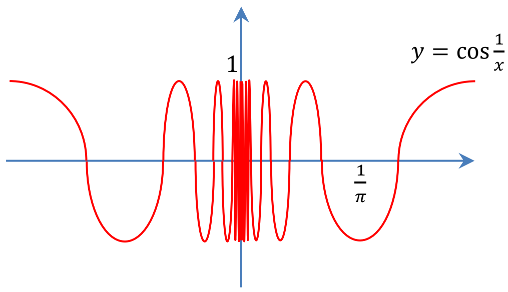
$$\lim_{x\to 0} \cos\frac{1}{x}$$

if it exists.

Solution:

We observe that

$$\cos\frac{1}{x} = \begin{cases} 1 & \text{for } x = \frac{1}{2\pi}, \frac{1}{4\pi}, \frac{1}{6\pi}, \frac{1}{8\pi}, \dots \\ -1 & \text{for } x = \frac{1}{\pi}, \frac{1}{3\pi}, \frac{1}{5\pi}, \frac{1}{7\pi}, \dots \end{cases}.$$



In particular, no matter how close x gets to 0 and $x \neq 0$, the value of $\cos\frac{1}{x}$ is sometimes -1 and sometimes 1, and is not always getting closer and closer to a unique real number. Therefore $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist.

The trend of $f(x)$ for x near a can be divided into two parts, either $x \to a$ from the "left" or $x \to a$ from the "right". These are called **one-sided limits**.

Definition 2.8 Let a be a real number.

⊙ Let f be a function that is well-defined on an open interval with right end-point a. If the number $f(x)$ gets closer and closer to a unique real number L when x gets closer and closer to a **and** $\boldsymbol{x < a}$, then L is called the "***left-hand limit*** of $f(x)$ as x tends to a". In symbols we write "$f(x) \to L$ as $x \to a^-$", or more compactly,

$$\lim_{x\to a^-} f(x) = L.$$

⊙ If f is well-defined on an open interval with left end-point a, then we define $\lim_{x\to a^+} f(x)$, i.e. the "***right-hand limit*** of $f(x)$ as x tends to a", in a similar manner (when x gets closer and closer to a **and** $\boldsymbol{x > a}$ instead).

Example 2.4 Let $f: \mathbb{R} \to \mathbb{R}$ be the Heaviside step function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

Then

$$\lim_{x\to 0^-} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 1.$$

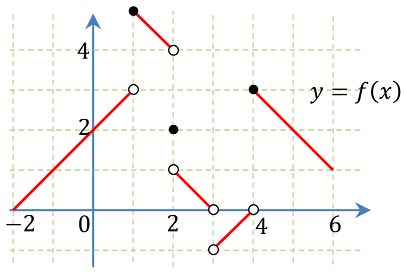
Example 2.7 Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function

$$f(x) = \cos\frac{1}{x}.$$

Then both

$$\lim_{x\to 0^-} f(x) \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) \qquad \text{do not exist.}$$

Example 2.9 The diagram on the right shows the graph of a function f. According to the graph, we have



$$\lim_{x\to 0^-} f(x) = 2, \qquad \lim_{x\to 0^+} f(x) = 2 \qquad \text{and} \qquad f(0) = 2;$$

$$\lim_{x\to 2^-} f(x) = 4, \qquad \lim_{x\to 2^+} f(x) = 1 \qquad \text{and} \qquad f(2) = 2;$$

$$\lim_{x\to 4^-} f(x) = 0, \qquad \lim_{x\to 4^+} f(x) = 3 \qquad \text{and} \qquad f(4) = 3.$$

Theorem 2.10 (Limit Existence Criterion) *Let a be a real number and f be a real-valued function which is well-defined on an open interval that contains a, except possibly at a. Then*

$$\lim_{x\to a} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Example 2.11 Let $f: \mathbb{R} \to \mathbb{R}$ be the absolute value function

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

Then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0,$$

so

$$\lim_{x\to 0} |x| = 0.$$

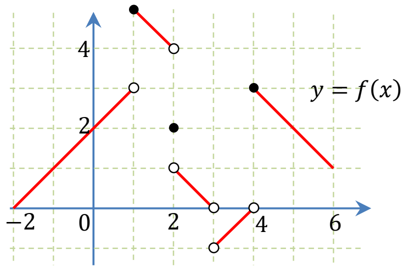
Remark 2.12 Let a be a real number and suppose that f is a **piecewise defined function** with different formulas for $x < a$ and for $x > a$. In order to find $\lim_{x \to a} f(x)$, we do the following:

(i) First find $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If any of them does not exist, neither does $\lim_{x \to a} f(x)$.

(ii) If both of them exist, then we next check whether $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$. If they are equal, then $\lim_{x \to a} f(x)$ exists and equals to the same number; if not, then $\lim_{x \to a} f(x)$ does not exist.

Note again that whether $\lim_{x \to a} f(x)$ exists is **<u>irrelevant</u>** to whether $f(a)$ is well-defined.

Example 2.13 The diagram below shows the graph of a function f.



(a) Using this graph, evaluate the one-sided limit $\lim_{x \to 0^-} f\big(f(x+2)\big)$.

(b) Does the limit $\lim_{x \to 0} f\big(f(x+2)\big)$ exist?

Solution:

(a) As $x \to 0^-$, we have $x + 2 \to 2^-$, so from the graph we see that $f(x+2) \to 4^+$. From the graph again, we see that $f\big(f(x+2)\big) \to 3$. Therefore

$$\lim_{x \to 0^-} f\big(f(x+2)\big) = 3.$$

(b) As $x \to 0^+$, we have $x + 2 \to 2^+$, so from the graph we see that $f(x+2) \to 1^-$. From the graph again, we see that $f\big(f(x+2)\big) \to 3$. Therefore

$$\lim_{x \to 0^+} f\big(f(x+2)\big) = 3.$$

Combining this with (a), we conclude that $\lim_{x \to 0} f\big(f(x+2)\big)$ exists and is also 3.

2. Rules and techniques of computing limits

Here are some more properties of limits of functions, regarding the interactions with **arithmetic operations** of functions.

Theorem 2.14 *Let a and c be real numbers and let f and g be functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist as* ***finite real numbers****. Then*

(i) $\lim_{x\to a} c = c$

(ii) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$

(iv) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

(v) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided that* $\lim_{x\to a} g(x) \neq 0$.

(vi) $\lim_{x\to a}[f(x)]^c = \left[\lim_{x\to a} f(x)\right]^c$, *provided that both sides are well-defined.*

Similar statements also hold for one-sided limits.

Example 2.15 Evaluate the limit

$$\lim_{x\to 1}\left(3x^2 + \sqrt{x}\right)$$

and explain carefully which rules in Theorem 2.14 have been applied in each step.

Solution:

As observed in Example 2.1, we have $\lim_{x\to 1} x = 1$. Then,

- $\lim_{x\to 1} x^2 = \left(\lim_{x\to 1} x\right)^2 = 1^2 = 1$ by (vi), and so $\lim_{x\to 1} 3x^2 = 3\left(\lim_{x\to 1} x^2\right) = 3(1) = 3$ by (iii).

- Moreover, $\lim_{x\to 1} \sqrt{x} = \lim_{x\to 1} x^{\frac{1}{2}} = \left(\lim_{x\to 1} x\right)^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$ by (vi) again.

Therefore, we have

$$\lim_{x\to 1}\left(3x^2 + \sqrt{x}\right) = \lim_{x\to 1} 3x^2 + \lim_{x\to 1} \sqrt{x} = 3 + 1 = 4$$

by (ii).

Note that Theorem 2.14 does not apply when the conditions are not met.

Example 2.3 It is incorrect to say that

$$\lim_{x\to 2}\frac{1}{x-2}=\frac{\lim_{x\to 2}1}{\lim_{x\to 2}(x-2)}, \quad \times \qquad \text{or that} \qquad \lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{\lim_{x\to 2}(x^2-4)}{\lim_{x\to 2}(x-2)}. \quad \times$$

This is because $\lim_{x\to 2}(x-2)=0$; but when applying Theorem 2.14 (v) we need $\lim_{x\to a}g(x)\neq 0$.

Remark 2.16 The formulas in Theorem 2.14 hold only when the limits on both sides **all exist as real numbers**. In particular, if some limit on one side of the equality does not exist, we **cannot** conclude that the limit on the other side of the equality does not exist either.

Example 2.17 Let

$$f(x)=\frac{1}{x}, \qquad g(x)=-\frac{1}{x} \qquad \text{and} \qquad h(x)=x.$$

Then

- $\lim_{x\to 0}f(x)$ and $\lim_{x\to 0}g(x)$ do not exist, but $\lim_{x\to 0}[f(x)+g(x)]=\lim_{x\to 0}\left[\frac{1}{x}+\left(-\frac{1}{x}\right)\right]=\lim_{x\to 0}0=0$ still exists as a real number. In this situation, Theorem 2.14 (ii) is **not applicable**.
- $\lim_{x\to 0}f(x)$ does not exist, but $\lim_{x\to 0}[f(x)h(x)]=\lim_{x\to 0}\left(\frac{1}{x}\cdot x\right)=\lim_{x\to 0}1=1$ still exists as a real number. In this situation, Theorem 2.14 (iv) is **not applicable**.

Example 2.18 Suppose that

$$\lim_{x\to 2}\frac{f(x)-3}{x-2}=5.$$

Find $\lim_{x\to 2}f(x)$.

Solution:

For every number $x\neq 2$ in the domain of f, we have $f(x)=\frac{f(x)-3}{x-2}\cdot(x-2)+3$. Therefore

$$\begin{aligned}\lim_{x\to 2}f(x)&=\lim_{x\to 2}\left[\frac{f(x)-3}{x-2}\cdot(x-2)+3\right]\\&=\lim_{x\to 2}\frac{f(x)-3}{x-2}\cdot\lim_{x\to 2}(x-2)+\lim_{x\to a}3 \qquad \text{(since all three limits exist)}\\&=5\cdot 0+3\\&=3.\end{aligned}$$

The **elementary functions** introduced in chapter 1 are nice, in the sense that we can **directly substitute** the value of an elementary function when computing its limit at a point in its domain.

> **Theorem 2.19** *Let f be a polynomial, a rational function, a power function, an exponential function, a logarithmic function, a trigonometric function or an inverse trigonometric function. Then for every a in the interior of the domain of f, we have*
>
> $$\lim_{x\to a} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

Example 2.20 From Theorem 2.19 we can deduce that

$$\lim_{x\to 1}\frac{x^2+3x+1}{4x+1} = \frac{1^2+3(1)+1}{4(1)+1} = 1, \qquad \lim_{x\to 1}\sin x = \sin 1, \qquad \lim_{x\to 1} e^x = e^1 = e,$$

because 1 is **in the interior of the domain** of each of the above elementary functions. However, each of the following is incorrect!

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0}, \ \times \qquad \lim_{x\to \frac{\pi}{2}}\tan x = \tan\frac{\pi}{2}, \ \times \qquad \lim_{x\to 0}\ln x = \ln 0. \ \times$$

Example 2.21 Let a be a real number, and let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \pi a \cos x + 2 & \text{if } x < \pi \\ a^2 x^2 & \text{if } x \geq \pi \end{cases}.$$

Find all the value(s) of a such that $\lim_{x\to\pi} f(x)$ exists.

Solution:

From the given definition of f, we have

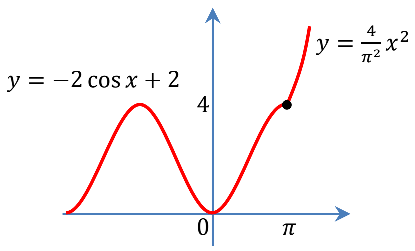
$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-}(\pi a\cos x + 2) = \pi a\cos\pi + 2 = -\pi a + 2,$$

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} a^2x^2 = a^2\pi^2.$$

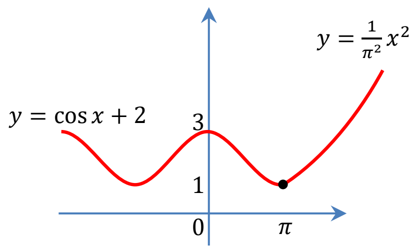
In order that $\lim_{x\to\pi} f(x)$ exists, we need $\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x)$; so

$$\begin{aligned} & -\pi a + 2 = a^2\pi^2 \\ \Rightarrow \quad & \pi^2 a^2 + \pi a - 2 = 0 \\ \Rightarrow \quad & (\pi a + 2)(\pi a - 1) = 0 \end{aligned}$$

which gives $a = -\frac{2}{\pi}$ or $a = \frac{1}{\pi}$.



when $a = -\frac{2}{\pi}$



when $a = \frac{1}{\pi}$

When we try to compute limits by direct substitution, we sometimes encounter an **indeterminate form** like $\frac{0}{0}$. This may happen when the limit is not taken at a point in the domain of the function (cf. Example 2.20). **Factorization** and **rationalization** are two useful methods to deal with these indeterminate forms. The identities

$$a^2-b^2=(a-b)(a+b)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$$

are often useful.

Example 2.22 Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 2}\frac{\sqrt{2+x}-2}{x-2}$ (b) $\displaystyle\lim_{x\to 0}\frac{4^x+4^{-x}-2}{4^x-4^{-x}}$ (c) $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$

Solution:

(a) [We aim to cancel the factor $(x-2)$ in the denominator.]

$$\lim_{x\to 2}\frac{\sqrt{2+x}-2}{x-2}=\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\cdot\frac{\sqrt{2+x}+2}{\sqrt{2+x}+2}\right)=\lim_{x\to 2}\frac{\left(\sqrt{2+x}\right)^2-2^2}{(x-2)\left(\sqrt{2+x}+2\right)}$$

$$=\lim_{x\to 2}\frac{x-2}{(x-2)\left(\sqrt{2+x}+2\right)}=\lim_{x\to 2}\frac{1}{\sqrt{2+x}+2}=\frac{1}{\sqrt{2+2}+2}=\frac{1}{4}$$

(b) [We aim to cancel a factor that makes the denominator zero.]

$$\lim_{x\to 0}\frac{4^x+4^{-x}-2}{4^x-4^{-x}}=\lim_{x\to 0}\frac{(2^x)^2+(2^{-x})^2-2(2^x)(2^{-x})}{(2^x)^2-(2^{-x})^2}$$

$$=\lim_{x\to 0}\frac{(2^x-2^{-x})^2}{(2^x-2^{-x})(2^x+2^{-x})}$$

$$=\lim_{x\to 0}\frac{2^x-2^{-x}}{2^x+2^{-x}}=\frac{2^0-2^{-0}}{2^0+2^{-0}}=0$$

(c) [We aim to cancel the factor $(x-1)$.]

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}=\lim_{x\to 1}\left[\frac{\sqrt[3]{x}-1}{x-1}\cdot\frac{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}\right]=\lim_{x\to 1}\frac{x-1}{(x-1)\left[\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1\right]}$$

$$=\lim_{x\to 1}\frac{1}{\left(\sqrt[3]{x}\right)^2+\sqrt[3]{x}+1}=\frac{1}{\left(\sqrt[3]{1}\right)^2+\sqrt[3]{1}+1}=\frac{1}{3}$$

We may apply **changes of variables** when computing limits, which sometimes change the limit to a form that we are more familiar with. We have already applied this idea in Example 2.13 when we computed a limit using the graph of the function.

Theorem 2.23 *Let a and b be real numbers and let f and g be functions. If g is a non-constant elementary function such that $\lim_{x\to a} g(x) = b$, then one can apply the change of variable $t = g(x)$ to obtain*

$$\lim_{x\to a} f\big(g(x)\big) = \lim_{g(x)\to b} f\big(g(x)\big) = \lim_{t\to b} f(t)\,,$$

provided that the last limit exists. A similar statement also holds for one-sided limits.

Example 2.22 (c) Evaluate the limit

$$\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x - 1}.$$

Solution:

This time we use the change of variable $t = \sqrt[3]{x}$; then $x = t^3$. We then have $t \to 1$ as $x \to 1$, so

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{x - 1} &= \lim_{t\to 1} \frac{t - 1}{t^3 - 1} = \lim_{t\to 1} \frac{t - 1}{(t-1)(t^2 + t + 1)} \\ &= \lim_{t\to 1} \frac{1}{t^2 + t + 1} = \frac{1}{1^2 + 1 + 1} = \frac{1}{3}.\end{aligned}$$

Another useful tool for computing limits is the following **Squeeze Theorem** (also known as Sandwich Theorem), which follows from the following lemma.

Lemma 2.24 *Let a be a real number and let f and g be functions such that both limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. If $f(x) \le g(x)$ for every x near a, except possibly at a, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)\,.$$

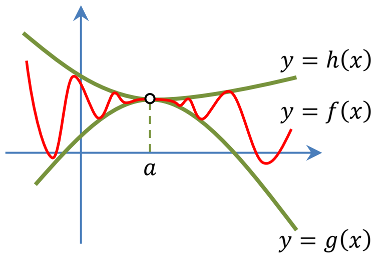
A similar statement also holds for one-sided limits.

Theorem 2.25 (Squeeze Theorem / Sandwich Theorem) *Let a be a real number and let f, g and h be functions. If $g(x) \leq f(x) \leq h(x)$ for every x near a except possibly at a, and if*

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x),$$

then $\lim_{x \to a} f(x)$ exists and also equals to the same limit. A similar statement also holds for one-sided limits.

Squeeze Theorem is particularly useful when we can bound a **complicated target function** by two **simpler functions** which have the **same limit**, as illustrated in the diagram below.



The inequalities

$$-1 \leq \sin x \leq 1 \qquad \text{and} \qquad -1 \leq \cos x \leq 1,$$

which hold for every real number x, are often useful.

Example 2.26 Evaluate the limit

$$\lim_{x \to 0} x \sin \frac{1}{x^2},$$

if it exists.

Solution:

Noting that $-1 \leq \sin \frac{1}{x^2} \leq 1$ for every $x \neq 0$, we have

$$-x \leq x \sin \frac{1}{x^2} \leq x \quad \text{for } x > 0, \qquad \text{and} \qquad x \leq x \sin \frac{1}{x^2} \leq -x \quad \text{for } x < 0.$$

Combining both cases, we always have

$$-|x| \leq x \sin \frac{1}{x^2} \leq |x| \qquad \text{for every } x \neq 0.$$

Since $\lim_{x \to 0} -|x| = 0$ and $\lim_{x \to 0} |x| = 0$ (cf. Example 2.11), by Squeeze Theorem we have

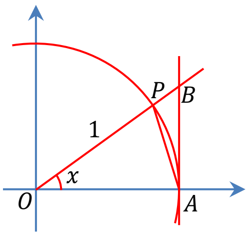
$$\lim_{x \to 0} x \sin \frac{1}{x^2} = 0.$$

The following is an **important limit about the sine function**.

Theorem 2.27

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Proof. We first try to evaluate the right-hand limit $\lim_{x\to 0^+}\frac{\sin x}{x}$. Suppose that $0<x<\frac{\pi}{2}$, and in the coordinate plane we let $O=(0,0)$, $P=(\cos x,\sin x)$ and $A=(1,0)$. Also let B be the point of intersection of the line OP and the line $x=1$, so that $B=(1,\tan x)$. The points P and B both lie in the first quadrant, as shown in the diagram below.



Now it is clear that

$$\text{Area of } \Delta OPA \le \text{Area of sector } OPA \le \text{Area of } \Delta OAB,$$

i.e.

$$\frac{1}{2}\sin x\le\frac{1}{2}x\le\frac{1}{2}\tan x.$$

Since $0<x<\frac{\pi}{2}$, we have $\cos x>0$, so we may rearrange the above inequality to get

$$\cos x\le\frac{\sin x}{x}\le 1.$$

Now $\lim_{x\to 0^+}\cos x=\cos 0=1$ and $\lim_{x\to 0^+}1=1$, so by Squeeze Theorem we have

$$\lim_{x\to 0^+}\frac{\sin x}{x}=1.$$

On the other hand, with the change of variables $t=-x$ we also have

$$\lim_{x\to 0^-}\frac{\sin x}{x}=\lim_{t\to 0^+}\frac{\sin(-t)}{-t}=\lim_{t\to 0^+}\frac{\sin t}{t}=1.$$

Therefore $\lim_{x\to 0}\frac{\sin x}{x}$ exists and also equals to 1. ■

The fact that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

is useful when handling limits with $\frac{0}{0}$ indeterminate forms that involve trigonometric functions.

Example 2.28 Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}$

(b) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$

(c) $\displaystyle\lim_{x\to \pi}\frac{\sin x}{x-\pi}$

(d) $\displaystyle\lim_{x\to a}\frac{\sin x-\sin a}{x-a}$, where a is a real number.

Solution:

(a)

$$\begin{aligned}\lim_{x\to 0}\frac{\tan 5x}{\sin 3x}&=\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{3x}{\sin 3x}\cdot\frac{5}{3\cos 5x}\right)\\&=\lim_{x\to 0}\frac{\sin 5x}{5x}\cdot\lim_{x\to 0}\frac{3x}{\sin 3x}\cdot\lim_{x\to 0}\frac{5}{3\cos 5x}\\&=1\cdot 1\cdot\frac{5}{3\cdot 1}=\frac{5}{3}\end{aligned}$$

As $x\to 0$, we also have $5x\to 0$ and $3x\to 0$.

(b)

As $x\to 0$, we also have $x/2\to 0$.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^2=\frac{1}{2}\cdot 1^2=\frac{1}{2}$$

(c) With the change of variable $t=x-\pi$, we have

$$\lim_{x\to \pi}\frac{\sin x}{x-\pi}=\lim_{t\to 0}\frac{\sin(t+\pi)}{t}=\lim_{t\to 0}\left(-\frac{\sin t}{t}\right)=-1.$$

(d) Recall the sum-to-product formula $\sin x-\sin a=2\cos\frac{x+a}{2}\sin\frac{x-a}{2}$. Therefore

$$\begin{aligned}\lim_{x\to a}\frac{\sin x-\sin a}{x-a}&=\lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}=\lim_{x\to a}\left(\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cos\frac{x+a}{2}\right)\\&=\lim_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\lim_{x\to a}\cos\frac{x+a}{2}=1\cdot\cos\frac{a+a}{2}=\cos a.\end{aligned}$$

As $x\to a$, we have $\frac{x-a}{2}\to 0$.

Note that this limit includes (c) as a special case when $a=\pi$.

3. Infinite limits and limits at infinity

The concept of limits can also be extended to the case **at infinity**.

Definition 2.29 Let f be a function.

- Suppose that f is well-defined on an open interval unbounded on the right. If $f(x)$ gets closer and closer to a unique real number L when x gets larger and larger, then L is called the "***limit*** of $f(x)$ as x tends to $+\infty$". In symbols we write "$f(x) \to L$ as $x \to +\infty$", or more compactly,

$$\lim_{x \to +\infty} f(x) = L.$$

- If f is well-defined on an open interval unbounded on the left, then we define $\lim_{x \to -\infty} f(x)$, i.e. the "***limit*** of $f(x)$ as x tends to $-\infty$", in a similar manner (when $-x$ gets larger and larger instead).

Remark 2.30 The arithmetic operations (Theorem 2.14), change of variables (Theorem 2.23), Lemma 2.24 and Squeeze Theorem (Theorem 2.25) also hold for limits at infinity.

On the other hand, we also allow the limit itself to be **infinite**.

Definition 2.31 Let a be a real number and f be a function which is well-defined on an open interval that contains a, except possibly at a.

- We say that the ***limit*** of $f(x)$ as x tends to a is "$+\infty$" if the number $f(x)$ gets arbitrarily large, and keeps getting larger and larger when x gets closer and closer to a **but** $\boldsymbol{x \neq a}$. In symbols we write "$f(x) \to +\infty$ as $x \to a$", or more compactly,

$$\lim_{x \to a} f(x) = +\infty.$$

- We say that the ***limit*** is "$-\infty$" and write $\lim_{x \to a} f(x) = -\infty$ if $\lim_{x \to a} -f(x) = +\infty$.

Remark 2.32 We define one-sided infinite limits in a similar manner as above, except that "$\boldsymbol{x \neq a}$" is modified to "$\boldsymbol{x > a}$" for the right-hand limit and "$\boldsymbol{x < a}$" for the left-hand limit. We also extend the notion to **infinite limits at infinity** and define symbols like

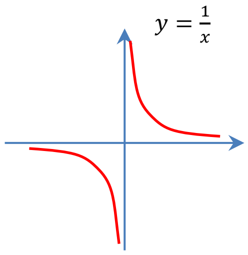
$$\lim_{x \to +\infty} f(x) = +\infty$$

and so on. The formulations are similar and are omitted here.

Having the notions of limits at infinity and infinite limits, we obtain the following useful limits of some elementary functions immediately from their graphs.

Example 2.33

$$\lim_{x\to 0^+}\frac{1}{x}=+\infty \qquad \lim_{x\to 0^-}\frac{1}{x}=-\infty \qquad \lim_{x\to +\infty}\frac{1}{x}=0 \qquad \lim_{x\to -\infty}\frac{1}{x}=0$$

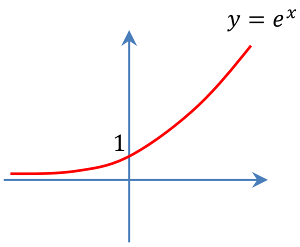


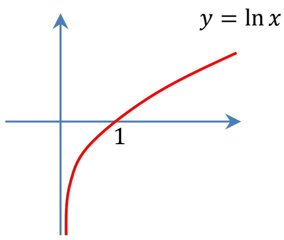
Example 2.34 For each $a>1$, we have

$$\lim_{x\to +\infty} a^x=+\infty \qquad \lim_{x\to -\infty} a^x=0 \qquad \lim_{x\to +\infty}\log_a x=+\infty \qquad \lim_{x\to 0^+}\log_a x=-\infty.$$

In particular,

$$\lim_{x\to +\infty} e^x=+\infty \qquad \lim_{x\to -\infty} e^x=0 \qquad \lim_{x\to +\infty}\ln x=+\infty \qquad \lim_{x\to 0^+}\ln x=-\infty.$$



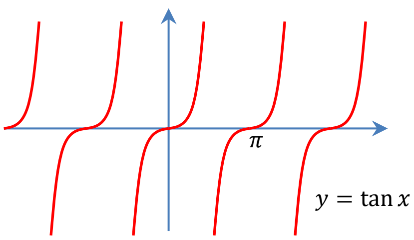


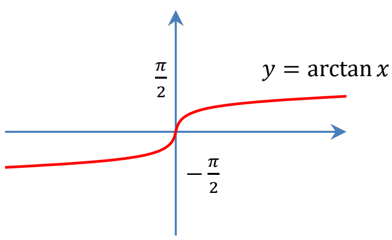
For each $a\in(0,1)$, we have

$$\lim_{x\to +\infty} a^x=0 \qquad \lim_{x\to -\infty} a^x=+\infty \qquad \lim_{x\to +\infty}\log_a x=-\infty \qquad \lim_{x\to 0^+}\log_a x=+\infty.$$

Example 2.35

$$\lim_{x\to (\pi/2)^-}\tan x=+\infty \qquad \lim_{x\to (-\pi/2)^+}\tan x=-\infty \qquad \lim_{x\to +\infty}\arctan x=\frac{\pi}{2} \qquad \lim_{x\to -\infty}\arctan x=-\frac{\pi}{2}$$





Example 2.36 Since $\lim_{x\to 0^-} \frac{1}{x^2} = +\infty$ and $\lim_{x\to 0^+} \frac{1}{x^2} = +\infty$, we have

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

Example 2.37 Consider the limit

$$\lim_{x\to +\infty} \sin x\,.$$

We observe that

$$\sin x = \begin{cases} 0 & \text{for } x = \pi, 2\pi, 3\pi, 4\pi, \dots \\ 1 & \text{for } x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots \end{cases}.$$

So no matter how large x becomes, $\sin x$ is sometimes 0 and sometimes 1, and is not always getting closer and closer to a unique real number. Therefore $\lim_{x\to +\infty} \sin x$ **does not exist**.

Example 2.38 Evaluate the following limits:

(a) $\displaystyle\lim_{x\to +\infty} \frac{6x^3 - 7x + 1}{-2x^3 + x^2}$

(b) $\displaystyle\lim_{x\to +\infty} \frac{x^3 + 1}{1 - 2x^2}$

(c) $\displaystyle\lim_{x\to 0^-} \frac{1}{3 + e^{\frac{1}{x}}}$

(d) $\displaystyle\lim_{x\to -\infty} \left(\sqrt{x^2 - 2x + 3} + x\right)$

Solution:

(a)

$$\lim_{x\to +\infty} \frac{6x^3 - 7x + 1}{-2x^3 + x^2} = \lim_{x\to +\infty} \frac{6 - \frac{7}{x^2} + \frac{1}{x^3}}{-2 + \frac{1}{x}} = \frac{6 - 0 + 0}{-2 + 0} = -3$$

(b) With the change of variable $t = \frac{1}{x}$, we have

$$\lim_{x\to +\infty} \frac{x^3 + 1}{1 - 2x^2} = \lim_{x\to +\infty} \frac{1 + \frac{1}{x^3}}{\frac{1}{x^3} - \frac{2}{x}} = \lim_{t\to 0^+} \frac{1 + t^3}{t^3 - 2t} = \lim_{t\to 0^+} \frac{\overbrace{1 + t^3}^{\to 1}}{\underbrace{t}_{\to 0^+}\underbrace{(t^2 - 2)}_{\to -2}} = -\infty.$$

(c) With the change of variable $t = \frac{1}{x}$, we have $\lim_{x\to 0^-} e^{\frac{1}{x}} = \lim_{t\to -\infty} e^t = 0$. So,

$$\lim_{x\to 0^-} \frac{1}{3+e^{\frac{1}{x}}} = \frac{1}{3+\lim_{x\to 0^-} e^{\frac{1}{x}}} = \frac{1}{3+0} = \frac{1}{3}.$$

(d)

$$\lim_{x\to -\infty}\left(\sqrt{x^2-2x+3}+x\right) = \lim_{x\to -\infty}\left[\left(\sqrt{x^2-2x+3}+x\right)\cdot\frac{\sqrt{x^2-2x+3}-x}{\sqrt{x^2-2x+3}-x}\right]$$

$$= \lim_{x\to -\infty}\frac{(x^2-2x+3)-x^2}{\sqrt{x^2-2x+3}-x} = \lim_{x\to -\infty}\frac{-2x+3}{-x\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}-x}$$

$\sqrt{x^2} = -x$ for $x<0$.

$$= \lim_{x\to -\infty}\frac{-2+3/x}{-\sqrt{1-2/x+3/x^2}-1} = \frac{-2+0}{-\sqrt{1-0+0}-1} = 1$$

Remark 2.39 When computing the limit of an expression, we should **NEVER** "replace part of the expression by its limit without handling the rest". Although sometimes one may still arrive at the correct answer, writing

$$\lim_{x\to -\infty}\frac{-2x+3}{-x\sqrt{1-2/x+3/x^2}-x} = \lim_{x\to -\infty}\frac{-2x+3}{-x\sqrt{1-0+0}-x} = \lim_{x\to -\infty}\frac{-2x+3}{-2x} = 1 \quad \times$$

is as **logically INCORRECT** as writing

$$\lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0}\frac{\lim_{x\to 0} x}{x} = \lim_{x\to 0}\frac{0}{x} = \lim_{x\to 0} 0 = 0. \quad \times$$

The following illustrates how to apply the **Squeeze Theorem** for a **limit at infinity**.

Example 2.40 Evaluate the limit

$$\lim_{x\to +\infty}\frac{\cos x}{e^x}$$

if it exists.

Solution:
Noting that $-1 \le \cos x \le 1$ for every real number x, we have

$$-\frac{1}{e^x} \le \frac{\cos x}{e^x} \le \frac{1}{e^x} \qquad \text{for every real number } x.$$

Since $\lim_{x\to +\infty} -\frac{1}{e^x} = 0$ and $\lim_{x\to +\infty}\frac{1}{e^x} = 0$, by Squeeze Theorem we have

$$\lim_{x\to +\infty}\frac{\cos x}{e^x} = 0.$$

An **asymptote** of the graph of a function is a line that approximates the trend of the function.

Definition 2.41 Let m and c be real numbers and let f be a function.

(i) If

$$\lim_{x\to c^-} f(x) = +\infty \text{ or } -\infty \qquad \text{or} \qquad \lim_{x\to c^+} f(x) = +\infty \text{ or } -\infty,$$

then the line $x = c$ is called a ***vertical asymptote*** of the graph of f.

(ii) If

$$\lim_{x\to+\infty} [f(x) - (mx + c)] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} [f(x) - (mx + c)] = 0,$$

then the line $y = mx + c$ is called a ***horizontal asymptote*** of the graph of f when $m = 0$, or a ***slant asymptote*** of the graph of f when $m \neq 0$.

Example 2.33 – 2.35 Here are some immediate examples of asymptotes of graphs.

⊙ The graph of the function

$$f(x) = \frac{1}{x}$$

has a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$.

⊙ The graph of the function

$$f(x) = e^x$$

has a horizontal asymptote $y = 0$ and has no vertical or slant asymptotes.

⊙ The graph of the function

$$f(x) = \ln x$$

has a vertical asymptote $x = 0$ and has no horizontal or slant asymptotes.

⊙ The graph of the function

$$f(x) = \tan x$$

has many vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, …, $x = -\frac{\pi}{2}$, $x = -\frac{3\pi}{2}$, $x = -\frac{5\pi}{2}$, …, and has no horizontal or slant asymptotes.

⊙ The graph of the function

$$f(x) = \arctan x$$

has two horizontal asymptotes $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$, and has no vertical or slant asymptotes.

Example 2.42 Let f be the function defined by the formula

$$f(x) = \frac{2x^3 - x^2 - 4x - 4}{x^2 - 4}.$$

Find all the asymptotes of the graph of f.

Solution:

(i) **Vertical asymptotes**: We first note that the denominator of f can be factorized as

$$x^2 - 4 = (x - 2)(x + 2).$$

For every $a \neq -2$ and $a \neq 2$, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{2x^3 - x^2 - 4x - 4}{(x-2)(x+2)} = \frac{2a^3 - a^2 - 4a - 4}{(a-2)(a+2)}$$

which is finite. So $x = a$ is not a vertical asymptote of the graph of f for any of these a's. It remains to consider the two vertical lines $x = -2$ and $x = 2$.

⊙ As $x \to -2$, the numerator of f tends to $2(-2)^3 - (-2)^2 - 4(-2) - 4 = -16$, so

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{\overbrace{2x^3 - x^2 - 4x - 4}^{\to -16}}{\underbrace{(x-2)}_{\to -4}\underbrace{(x+2)}_{\to 0^+}} = +\infty.$$

This one-sided limit is infinite, so $x = -2$ is a vertical asymptote of the graph of f.

⊙ As $x \to 2$, the numerator of f tends to $2(2)^3 - (2)^2 - 4(2) - 4 = 0$, so $(x - 2)$ is a factor of the numerator of f. Factorizing the numerator, we get

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{(x-2)(2x^2 + 3x + 2)}{(x-2)(x+2)} \\ &= \lim_{x\to 2} \frac{2x^2 + 3x + 2}{x + 2} \\ &= \frac{2(2)^2 + 3(2) + 2}{2 + 2} = 4\end{aligned}$$

which is finite. So $x = 2$ is not a vertical asymptote of the graph of f.

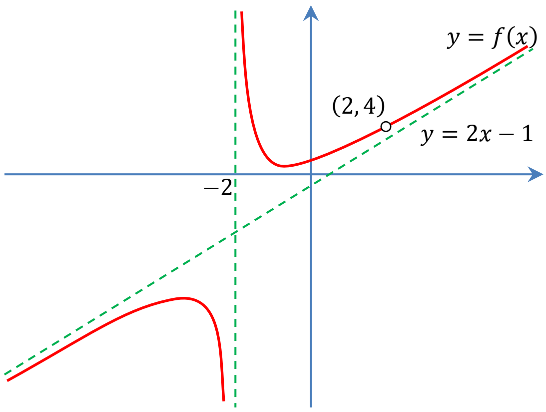
(ii) **Horizontal or slant asymptotes**: Using long division, the rational function f can be rewritten as

$$f(x) = \frac{2x^3 - x^2 - 4x - 4}{x^2 - 4} = 2x - 1 + \frac{4x - 8}{x^2 - 4}.$$

Thus we have

$$\lim_{x\to\pm\infty} [f(x) - (2x - 1)] = \lim_{x\to\pm\infty} \frac{4x - 8}{x^2 - 4} = \lim_{x\to\pm\infty} \frac{\frac{4}{x} - \frac{8}{x^2}}{1 - \frac{4}{x^2}} = \frac{0 - 0}{1 - 0} = 0,$$

and so the line $y = 2x - 1$ is a slant asymptote of the graph of f. The graph of f has no horizontal asymptotes.



In the previous example, we took advantage of the fact that f is a **rational function** when finding the numbers m and c for the horizontal or slant asymptotes. In general when f is **any kind of function**, if the graph of f has horizontal or slant asymptotes, then we can always find these asymptotes using the following result.

Lemma 2.43 *Let m and c be real numbers and let f be a function. If the line $y = mx + c$ is a horizontal or slant asymptote of the graph of f, then either*

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} \qquad \text{and} \qquad c = \lim_{x\to+\infty} [f(x) - mx],$$

or

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} \qquad \text{and} \qquad c = \lim_{x\to-\infty} [f(x) - mx].$$

Proof. The proof is left as an exercise for the reader (see Q16 of Problem Set 2). ■

Remark 2.44 To find the asymptotes of the graph of a function f, we need to do the following:

(i) First identify the **domain** of f.

(ii) For each **boundary point** a of the domain of f, we compute the one-sided limits

$$\lim_{x\to a^-} f(x) \qquad \text{and} \qquad \lim_{x\to a^+} f(x).$$

If **at least one of them is infinite**, then $x = a$ is a **vertical asymptote** of the graph of f.

(iii) If the domain of f is **unbounded** towards $+\infty$, then we compute the limit

$$\lim_{x\to+\infty} \frac{f(x)}{x}.$$

If this limit exists as a finite real number, then we call this real number m and proceed to compute the limit

$$\lim_{x\to+\infty} [f(x) - mx]$$

If this second limit also exists as a finite real number c, then the line $y = mx + c$ is a **horizontal or slant asymptote of** the graph of f. Do the same for $-\infty$ if the domain of f is unbounded towards $-\infty$.

Example 2.45 Let f be the function defined by the formula

$$f(x) = x\sin\frac{1}{x}.$$

Find all the asymptotes of the graph of f.

Solution:

The domain of f is $(-\infty, 0) \cup (0, +\infty)$.

⦿ The only boundary point of the domain is 0. We can check that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

by a Squeeze Theorem argument similar to Example 2.26. Since none of the one-sided limits is infinite, the graph of f has no vertical asymptotes.

⦿ Next we compute the limits

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \sin\frac{1}{x} = \sin 0 = 0,$$

and also

$$\lim_{x\to\pm\infty} [f(x) - 0x] = \lim_{x\to\pm\infty} x\sin\frac{1}{x} = \lim_{t\to 0^\pm} \frac{\sin t}{t} = 1$$

with a change of variable $t = \frac{1}{x}$. Therefore the line $y = 0x + 1$, or simply $y = 1$, is a horizontal asymptote of the graph of f. The graph of f has no slant asymptotes.

4. Continuity

We have seen in Theorem 2.19 that we can **directly substitute** the value of an elementary function when computing its limit at a point where it is well-defined. In general, functions having this property are called **continuous functions**.

Definition 2.46 Let I be an interval, let $f: I \to \mathbb{R}$ be a function and let $a \in I$ be a real number.

(i) If a is in the interior of I, then we say that f is ***continuous at*** $\boldsymbol{a}$ if

$$\lim_{x\to a} f(x) = f(a).$$

(ii) If a is the left end-point of I, then we say that f is ***continuous at*** $\boldsymbol{a}$ if

$$\lim_{x\to a^+} f(x) = f(a).$$

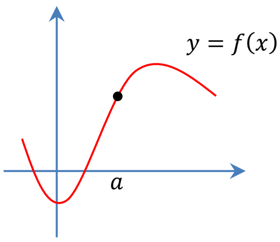
(iii) If a is the right end-point of I, then we say that f is ***continuous at*** $\boldsymbol{a}$ if

$$\lim_{x\to a^-} f(x) = f(a).$$

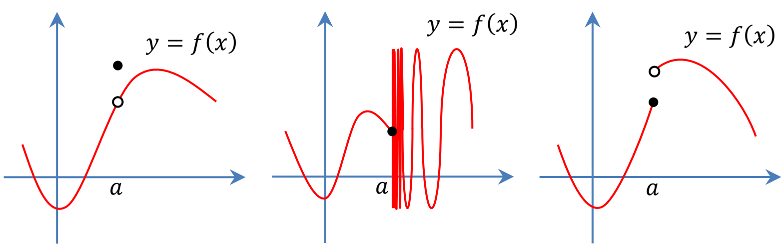
We say that f is ***discontinuous at*** $\boldsymbol{a}$ if it is not continuous at a.

Remark 2.47 According to the above definition, by saying that f is continuous at a we implicitly require that the following three statements all hold:

- f is well-defined at a (i.e. $f(a)$ is well-defined, or a is in the domain of f).
- The limit $\lim_{x\to a} f(x)$ (or the one-sided limit at an end-point) exists as a real number.
- $\lim_{x\to a} f(x) = f(a)$.



f is continuous at a



f is discontinuous at a

Definition 2.48 We say that a function f is ***continuous on an interval*** if it is continuous at every number in that interval. We say that f is ***continuous everywhere***, or simply f is ***continuous***, if it is continuous on its whole domain.

Intuitively speaking, a function f is **continuous at $\boldsymbol{a}$** if its graph does not "break" and has no "sudden jump" when going through the point $(a, f(a))$. f is **continuous on an interval** if its whole graph can be drawn **without lifting the pen away from the paper**.

There are many examples of continuous functions. For instance, restating Theorem 2.19 we have:

Theorem 2.49 *Polynomials, rational functions, power functions, exponential functions, logarithmic functions, trigonometric functions and inverse trigonometric functions are all continuous.*

Theorem 2.50 *The absolute value function is continuous.*

Proof. For every $a \neq 0$, the absolute value function (near a) is just a monomial, either x or $-x$; so it is continuous at every $a \neq 0$ by Theorem 2.19. Moreover, as in Example 2.11 we have

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0, \qquad \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0 \qquad \text{and} \qquad |0| = 0,$$

so the absolute value function is continuous at 0 also. ∎

Restating Theorem 2.14 we also have

Theorem 2.51 *Let f and g be continuous functions. Then $f+g$, $f-g$ and fg are all continuous, and $\frac{f}{g}$ is continuous at every number except at those a where $g(a) = 0$.*

Example 2.52 Let

$$f(x) = \begin{cases} a^2x^2 - 2a & \text{if } x \leq 1 \\ a\cos \pi x + 1 & \text{if } x > 1 \end{cases}.$$

Determine the possible value(s) of a such that f is continuous everywhere.

Solution:
On the intervals $(-\infty, 1)$ and $(1, +\infty)$, f is always continuous since it consists of sums of well-defined polynomials and trigonometric functions only. In order for f to be continuous at 1, we need to require that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1).$$

Since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (a^2x^2 - 2a) = a^2 - 2a,$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (a\cos \pi x + 1) = a\cos \pi + 1 = -a + 1,$$

$$f(1) = a^2(1)^2 - 2a = a^2 - 2a,$$

the requirement becomes

$$\begin{aligned} & a^2 - 2a = -a + 1 \\ \Rightarrow\quad & a^2 - a - 1 = 0 \\ \Rightarrow\quad & a = \frac{1+\sqrt{5}}{2} \quad \text{or} \quad a = \frac{1-\sqrt{5}}{2}. \end{aligned}$$

When computing limits, it is often convenient if we can move the limit sign through a function symbol. It turns out that this **swapping of the limit sign and the function symbol** is valid if the function is **continuous**. This result follows from Theorem 2.19 about change of variables.

Theorem 2.53 *Let a be a real number and f and g be functions. If $\lim_{x\to a} g(x) = b$ exists as a real number and if f is well-defined near this real number b and is continuous at b, then*

$$\lim_{x\to a} f\big(g(x)\big) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

A similar statement also holds for one-sided limits and for limits at infinity.

Example 2.54 Evaluate the limit $\lim_{x\to\frac{\pi}{2}} \tan(\sin x)$.

Solution:

Since $\lim_{x\to\frac{\pi}{2}} \sin x = \sin\frac{\pi}{2} = 1$ and the function $\tan$ is continuous at 1, we have

$$\lim_{x\to\frac{\pi}{2}} \tan(\sin x) = \tan\left(\lim_{x\to\frac{\pi}{2}} \sin x\right) = \tan 1.$$

From Theorem 2.53 we deduce the following, which leads to even more examples of continuous functions.

Theorem 2.55 *The composition of two continuous functions is continuous. In particular, if f is a continuous function, then the functions $|f|$, f^a, e^f, $\ln f$, $\cos f$, $\sin f$, $\arccos f$, $\arcsin f$, etc. are all continuous.*

There are two important properties of continuous functions of a real variable:

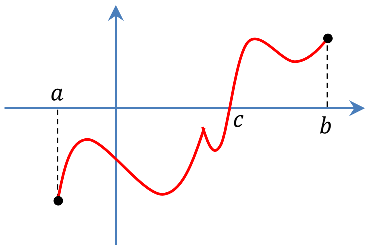
- ⦿ **Intermediate Value Theorem**
- ⦿ **Extreme Value Theorem**

Theorem 2.56 (Intermediate Value Theorem) *Let* $f: [a, b] \to \mathbb{R}$ *be a function which is continuous on a* ***bounded closed interval*** $[a, b]$. *If*

$$f(a)f(b) < 0$$

(one of $f(a)$ *or* $f(b)$ *is positive and the other is negative), then there exists* $c \in (a, b)$ *such that*

$$f(c) = 0.$$



Intermediate Value Theorem is often useful in showing the **existence of some point taking certain function values**, for example zeros of some functions or solutions to some equations.

Example 2.57 Let $f(x) = x^3 - 8x + 5$. Since f is a polynomial, it is continuous everywhere. Since

$$f(0) = 0^3 - 8(0) + 5 = 5 > 0 \qquad \text{and} \qquad f(1) = 1^3 - 8(1) + 5 = -2 < 0,$$

by Intermediate Value Theorem, we know that f has at least one root in the interval $(0, 1)$.

Example 2.57 Show that the polynomial

$$f(x) = x^3 - 8x + 5$$

has exactly three real roots.

Proof: [It is hard to find the three real roots exactly if we do not remember the "cubic formula". But just to **show the existence** of three real roots, we can simply use Intermediate Value Theorem.]

- ⦿ We have seen that $f(0) > 0$ and $f(1) < 0$. Moreover, we also have

  $$f(-100) < 0 \qquad \text{and} \qquad f(100) > 0.$$

  Since f is a polynomial, it is continuous everywhere. Applying **Intermediate Value Theorem** to f, we see that f has at least one root in $(-100, 0)$, at least one root in $(0, 1)$ and at least one root in $(1, 100)$. So f has at least three distinct real roots.
- ⦿ On the other hand, since f is a polynomial of degree 3, f has at most three real roots (cf. Theorem 1.19).

Therefore f has exactly three real roots. ■

The following definitions will help us understand the statement of the **Extreme Value Theorem**.

Definition 2.58 Let $A \subseteq \mathbb{R}$ be the domain of a function $f: A \to \mathbb{R}$, and let $c \in A$ be a real number in the domain of f.

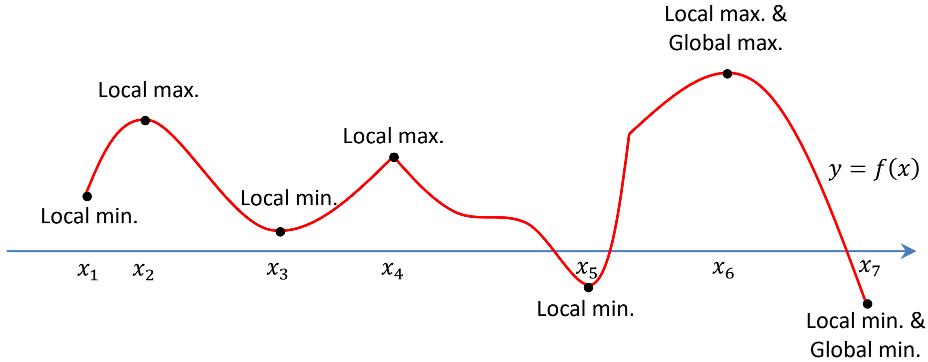
- If
$$f(x) \le f(c) \qquad \text{for every } x \in A,$$
then we say that f attains ***global maximum*** (or ***absolute maximum***) at c. The value $f(c)$ is called a ***global maximum value*** of f, and the corresponding point $(c, f(c))$ on the graph of f is called a ***global maximum point***.
- If
$$f(x) \ge f(c) \qquad \text{for every } x \in A,$$
then we say that f attains ***global minimum*** (or ***absolute minimum***) at c. The value $f(c)$ is called a ***global minimum value*** of f, and the corresponding point $(c, f(c))$ on the graph of f is called a ***global minimum point***.

Global maximum (*pl.* maxima) and global minimum (*pl.* minima) are collectively called ***global extremum*** (*pl.* extrema).

Definition 2.59 Let $A \subseteq \mathbb{R}$ be the domain of a function $f: A \to \mathbb{R}$, and let $c \in A$ be a real number in the domain of f.

- If $f(x) \le f(c)$ for every $x \in A$ that are near c, i.e. if there exists $\delta > 0$ such that
$$f(x) \le f(c) \qquad \text{for every } x \in (c-\delta, c+\delta) \cap A,$$
then we say that f attains ***local maximum*** (or ***relative maximum***) at c. The value $f(c)$ is called a ***local maximum value*** of f, and the corresponding point $(c, f(c))$ on the graph of f is called a ***local maximum point***.
- If $f(x) \ge f(c)$ for every $x \in A$ that are near c, i.e. if there exists $\delta > 0$ such that
$$f(x) \ge f(c) \qquad \text{for every } x \in (c-\delta, c+\delta) \cap A,$$
then we say that f attains ***local minimum*** (or ***relative minimum***) at c. The value $f(c)$ is called a ***local minimum value*** of f, and the corresponding point $(c, f(c))$ on the graph of f is called a ***local minimum point***.

Local maximum and local minimum are collectively called ***local extremum***.

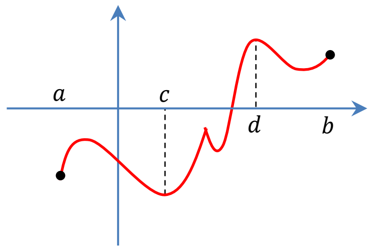


Theorem 2.60 (Extreme Value Theorem) *Let* $f: [a, b] \to \mathbb{R}$ *be a function which is continuous on a* ***bounded closed interval*** $[a, b]$*. Then* f *attains global maximum somewhere in* $[a, b]$ *and global minimum somewhere in* $[a, b]$*, i.e. there exist* $c, d \in [a, b]$ *such that*

$$f(c) \le f(x) \le f(d)$$

for every $x \in [a, b]$*.*

Remark 2.61 Graphically, the Extreme Value Theorem says that the graph of a continuous function on a bounded closed interval must have at least one **highest point** and one **lowest point**. Note that



- Extreme Value Theorem only asserts the **existence** of these global maxima and minima, but does not tell us how to find them. Later in chapter 4 we will learn how to find these points for some special class of continuous functions.
- This theorem is **<u>not</u>** true for a continuous function on an **open interval**. As a counter example, the function

$$f(x) = \tan x$$

is continuous on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but its graph has neither a highest point nor a lowest point because

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty \qquad \text{and} \qquad \lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty.$$

It is still **<u>not</u>** true even for a **bounded** continuous function on an **open interval**.

<u>Summary of Chapter 2</u>

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ The **informal definition of limits** of functions at a real number
- ✓ **One-sided limits**
- ✓ Evaluating limits of functions **graphically**

- ✓ Basic **arithmetic operations** on limits
- ✓ Limits of **elementary functions**
- ✓ Evaluating limits of functions using **factorization** or **rationalization**
- ✓ Evaluating limits of functions using a **change of variables**
- ✓ **Squeeze Theorem**
- ✓ $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

- ✓ **Limits at infinity**
- ✓ **Infinite limits**
- ✓ **Asymptotes** of graphs of functions: Vertical, horizontal and slant asymptotes

- ✓ Definition of **continuity** at a point and on an interval
- ✓ **Elementary functions are continuous** on their domains
- ✓ If f and g are continuous, then $f+g$, $f-g$, fg, $\frac{f}{g}$, $f\circ g$, $|f|$, f^c and f^{-1} are all continuous on their respective domains.
- ✓ If f is continuous, then $\lim\limits_{x\to a} f(g(x)) = f\left(\lim\limits_{x\to a} g(x)\right)$.
- ✓ **Intermediate Value Theorem**
- ✓ **Extreme Value Theorem**
- ✓ **Absolute extrema** and **relative extrema** of a function